Seismic Site Response (Site Amplification)

An Introduction to Shear Beam Analysis (Part I)

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(Initial version prepared in 2006 in collaboration with Drs. Liangcai He and Zhaohui Yang)

Source of seismic disturbance:

Earthquake **Fault** slips suddenly (sideways/vertical relative motion) sending a shear-dominated string of pulses that propagate away from the fault-plane and reach the ground surface.

How Earthquakes Happen

An aerial view of the San Andreas fault in the Carrizo Plain, Central California.

From: http://pubs.usgs.gov/gip/earthq1/how.html

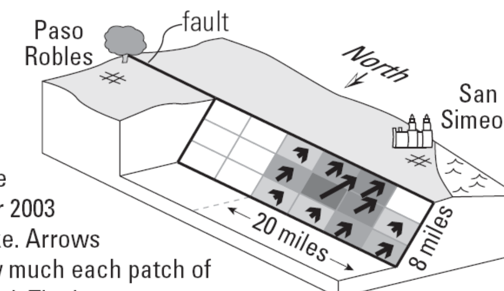
Plate tectonics: The cycle of earthquakes continues because plates motions continue.

A new fence is built straight across the fault at the boundary between two plates.

Over many years, plate motions cause strain to build up and deform the earth (and fence).

An earthquake is a sudden burst of motion that relieves the strain and causes shaking.

View into the Earth along the fault of the December 2003 earthquake. Arrows show how much each patch of fault moved. The largest movement was 9 feet. [Chen Ji, UC Santa Barbara]



white patches did not slip

From: http://pubs.usgs.gov/gip/2006/21/gip-21.pdf

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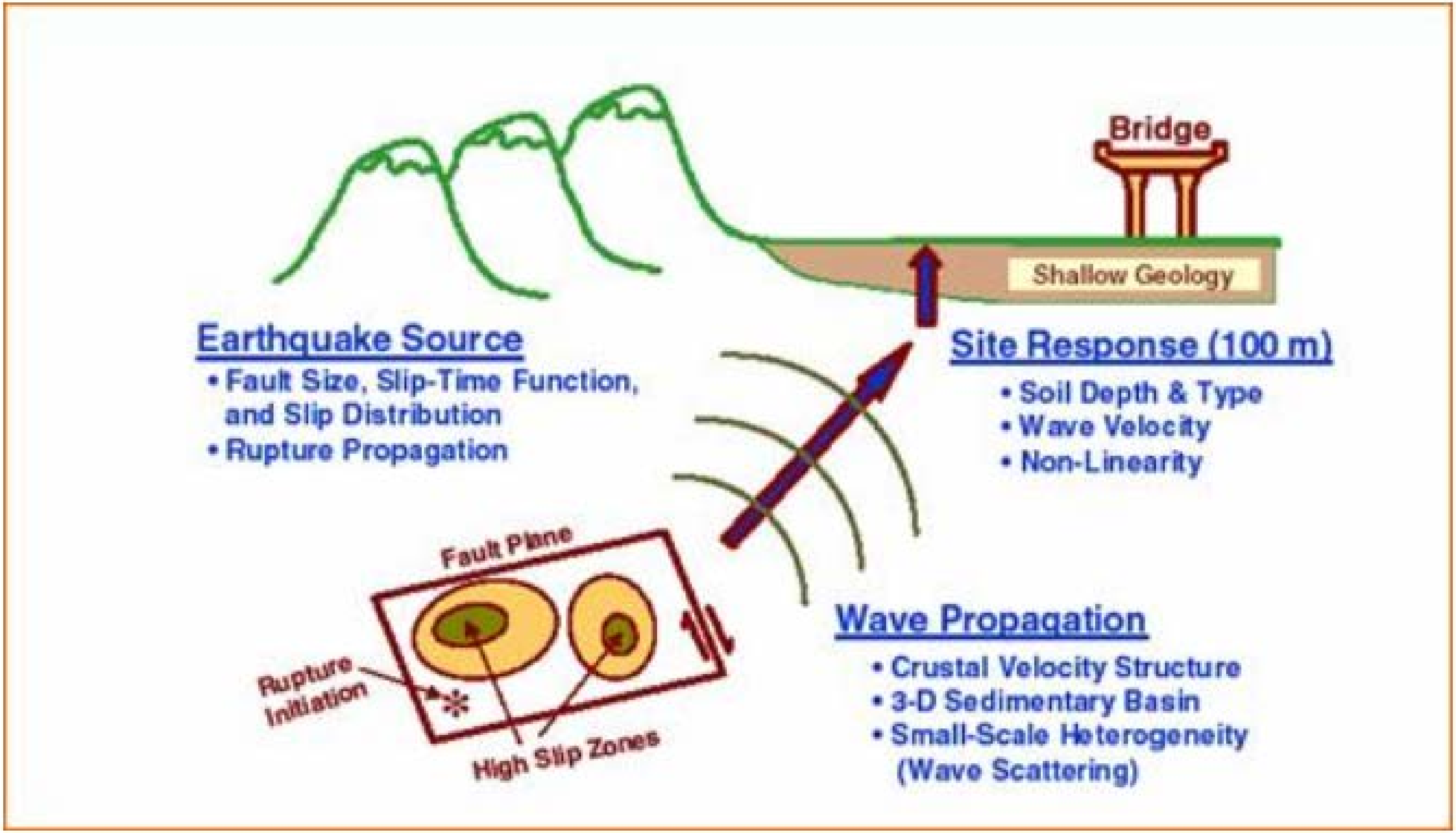
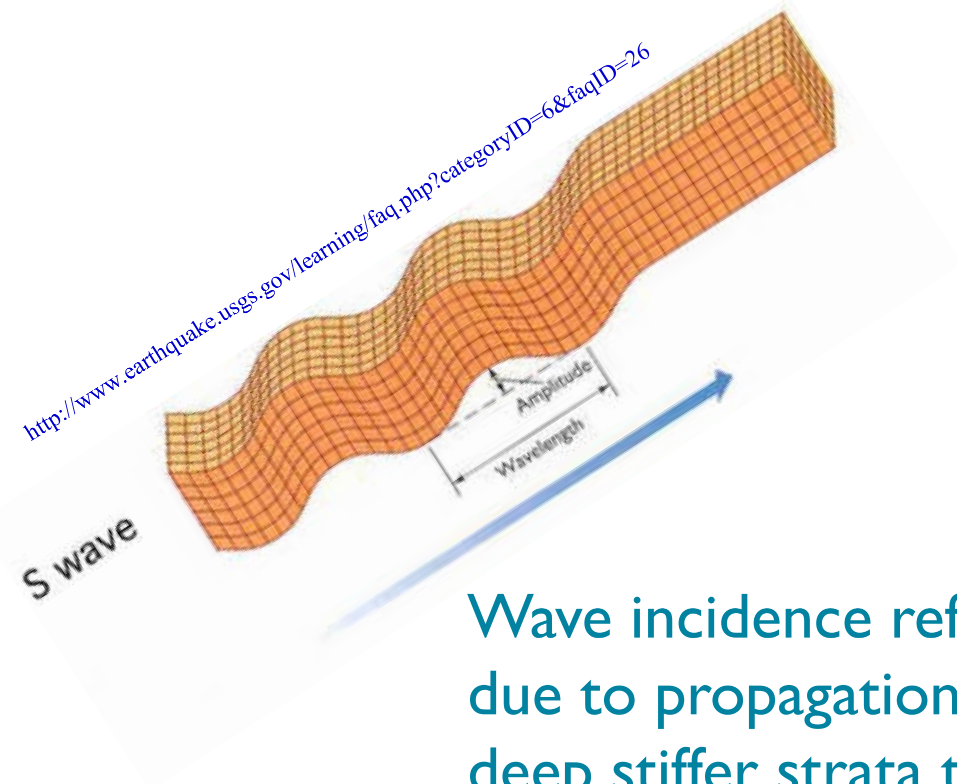
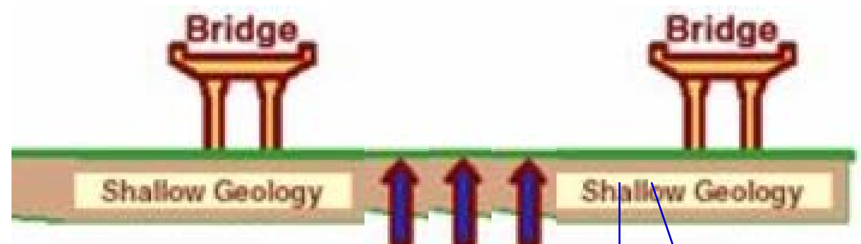


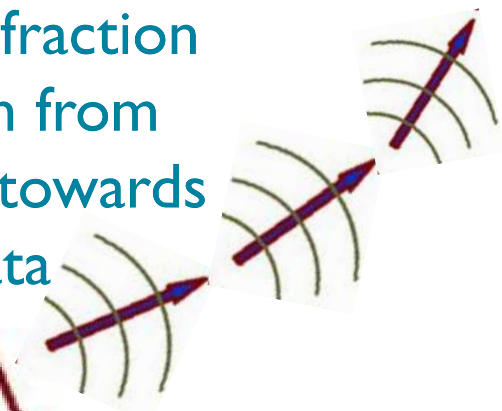
Illustration from: http://www.uky.edu/KGS/geologichazards/risks.htm

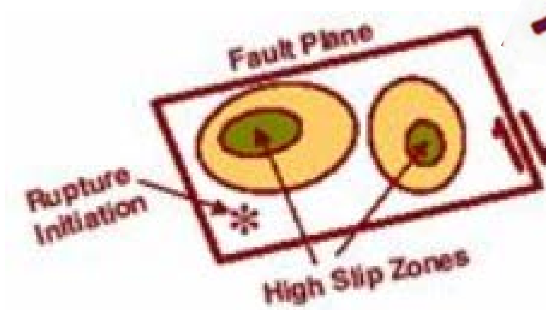
Shear (S) Wave

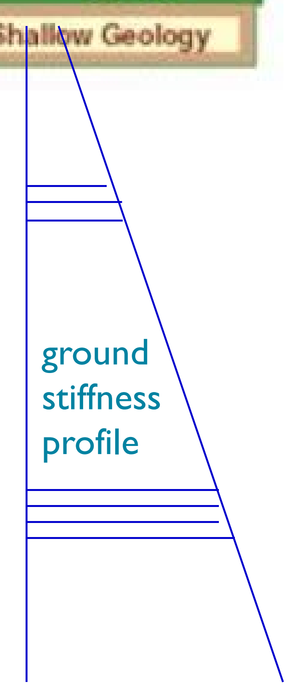




Wave incidence refraction due to propagation from deep stiffer strata towards shallow softer strata

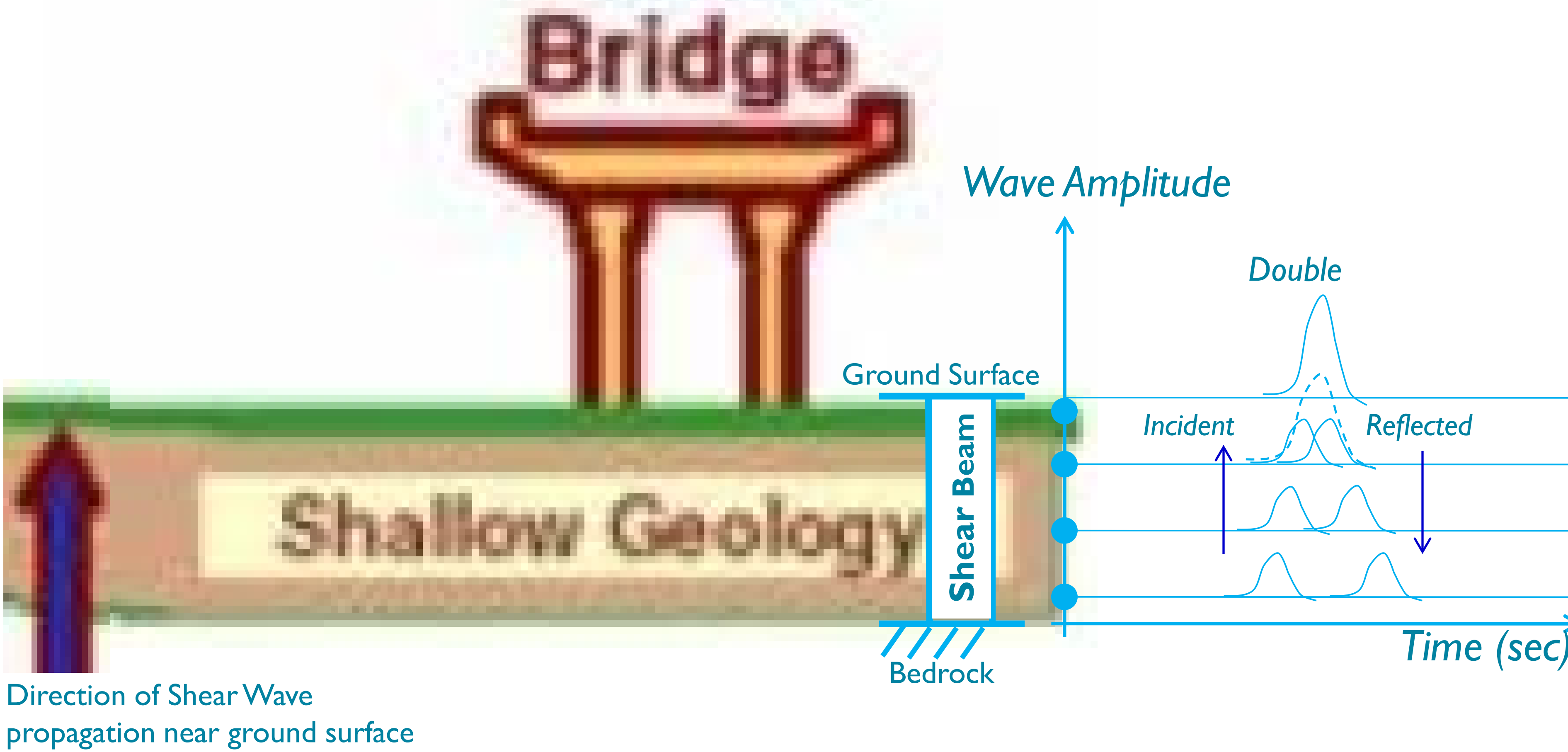






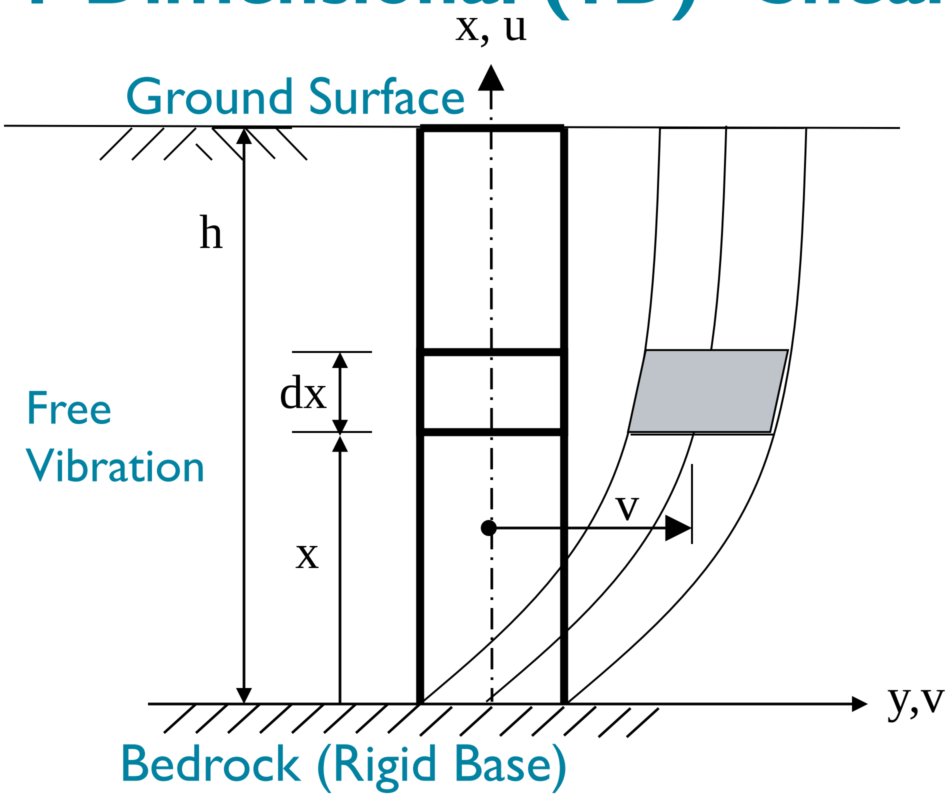
Near-surface shear-wave vertical incidence implies one-dimensional shear beam response

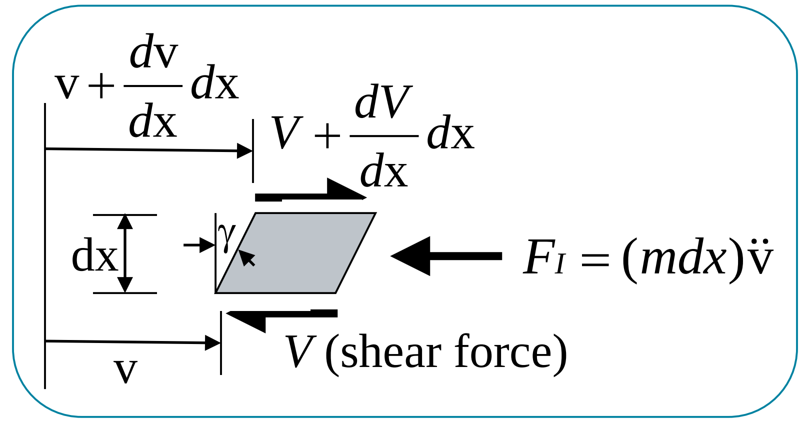
Site **Amplification** due to wave reflection at the ground surface



Schematic of amplification (sum of incident + reflected waves). An arbitrary train of actual seismic waves experiences similar phases of constructive wave amplification.

1-Dimensional (1D) "Shear Beam" Model





Forces acting on a representative element

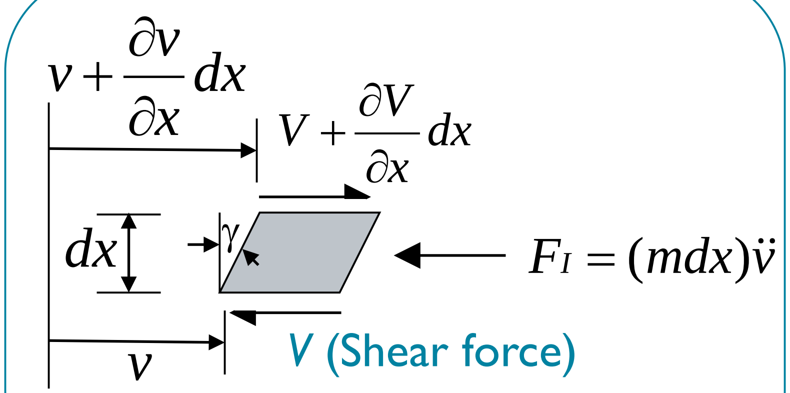
v = lateral displacement, $\gamma = dv/dx$ (shear strain), m = mass per unit height

V = shear force, F_I = inertia Force

- Continuous model useful for illustrating basic concepts
- Uniform shear across the section is assumed
- Represents seismic response of near-surface ground strata
- Represents first order approximation for tall framed structures

Equation of Motion

Shear Force $V = \tau A_r = G\gamma A_r = GA_r \dfrac{\partial v}{\partial x}$



τ : shear stress, and γ : shear strain $= \partial v / \partial x$
G : shear modulus (constant with depth)
A_r : reduced cross sectional area = k'A
k' : coefficient of shear (1.0 for soil shear beam)
A : total cross-sectional area

Inertial Force $F_I = mdx \dfrac{\partial^2 v}{\partial t^2} = \rho A dx \dfrac{\partial^2 v}{\partial t^2}$

m : mass per unit height
ρ : mass density

Σ *F=0* results in the Free Vibration Equation of Motion:

$$\rho A dx \frac{\partial^2 v}{\partial t^2} = \frac{\partial V}{\partial x} dx = GA_r \frac{\partial^2 v}{\partial x^2} dx \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2} = \frac{G}{\rho} \frac{\partial^2 v}{\partial x^2} = V_s^{\,2} \frac{\partial^2 v}{\partial x^2}$$

where $V_s = \sqrt{\dfrac{G}{\rho}}$ = shear wave velocity

Solution (by separation of variables)

$$v(x,t) = X(x)\,T(t) \quad \rightarrow \quad \frac{\ddot{T}}{T} = \frac{GA_r}{m}\frac{X''}{X} = -\omega^2$$

$$\frac{d^2T}{dt^2} + \omega^2 \cdot T(t) = 0 \quad \rightarrow \quad T(t) = A\sin\omega t + B\cos\omega t$$ (i.e., harmonic free-vibration)

$$\frac{d^2X}{dx^2} + \frac{m}{GA_r}\omega^2 X = 0 \quad \rightarrow \quad X(x) = A^*\sin\sqrt{\frac{m}{GA_r}}\omega x + B^*\cos\sqrt{\frac{m}{GA_r}}\omega x$$

Boundary Conditions

(i) at $x = 0$, $v(0,t) = 0$ "Fixed Base", so $B^* = 0$, or $X(x) = A^*\sin\sqrt{\frac{m}{GA_r}}\omega x$

(ii) at $x = h$, $V = GA_r\frac{\partial v}{\partial x} = 0$ "No shear at Ground Surface; stress-free"

$$X'(h) = 0 = A^*\sqrt{\frac{m}{GA_r}}\omega\cos\sqrt{\frac{m}{GA_r}}\omega h = 0 \rightarrow \cos\sqrt{\frac{m}{GA_r}}\omega h = 0 \rightarrow \sqrt{\frac{m}{GA_r}}\omega_n \cdot h = \left(\frac{2n-1}{2}\right)\pi, \quad n = 1, 2, 3, \ldots$$

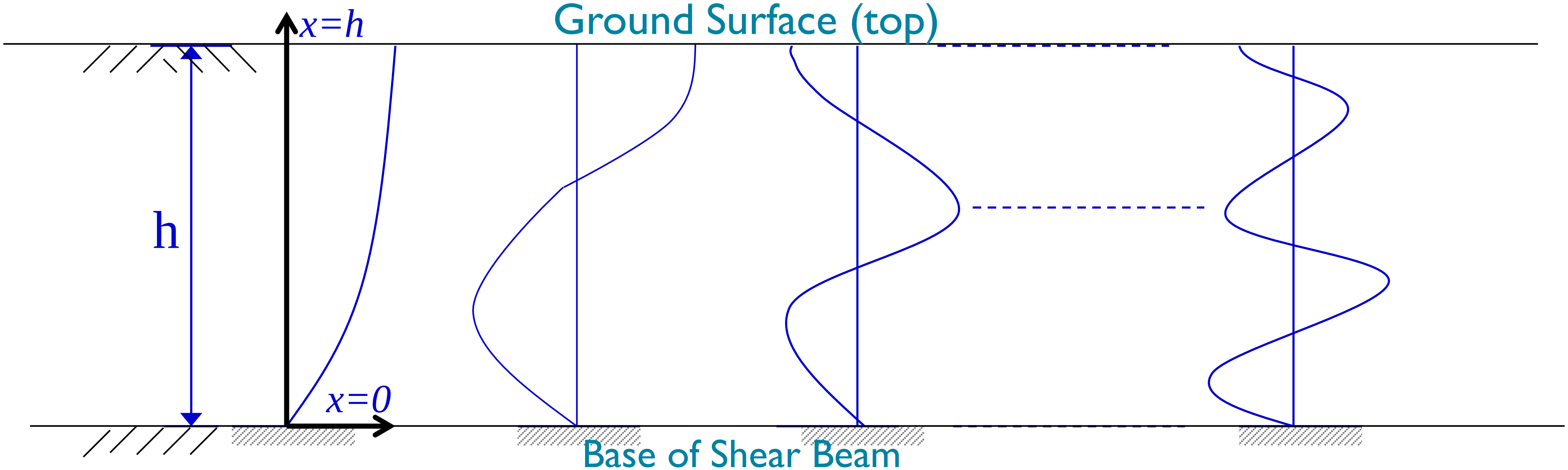
Frequencies

$$\omega_n = \sqrt{\frac{GA_r}{m}}\frac{(2n-1)\cdot\pi}{2h} = 0 \qquad \omega_n = \sqrt{\frac{G}{\rho}}\frac{(2n-1)\cdot\pi}{2h} = 0 \qquad \omega_n = V_s\frac{(2n-1)\cdot\pi}{2h} = 0 \quad n = 1, 2, 3, \ldots$$

$V_s = \sqrt{\frac{G}{\rho}}$, where V_s is the shear wave velocity, and $f_1 = V_s/4h$ (Hz), $f_2 = 3f_1$, $f_3 = 5f_1$

Mode Shapes

$$X_n(x) = \sin\frac{(2n-1)\pi x}{2h}, \qquad n = 1, 2, 3, ...$$



$\omega_1(x) = \frac{\pi}{2h}\sqrt{\frac{GAr}{m}}$	$\omega_2 = 3\omega_1$	$\omega_3 = 5\omega_1$	$\omega_n = \frac{(2n-1)\pi}{2h}\sqrt{\frac{GAr}{m}}$
$X_1(x) = \sin\frac{\pi x}{2h}$	$X_2(x) = \sin\frac{3\pi x}{2h}$	$X_3(x) = \sin\frac{5\pi x}{2h}$ -------	$X_n(x) = \sin\frac{(2n-1)\pi x}{2h}$

Notes:

1. The ratios of the natural frequencies go as 1, 3, 5, 7, …
2. Modes can be normalized such that $X_n(h) = \pm 1$

General Solution (Free Vibration)

$$v(x,t)=\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi x}{2h}(A_n\sin\omega_n t+B_n\cos\omega_n t)$$

Initial Conditions

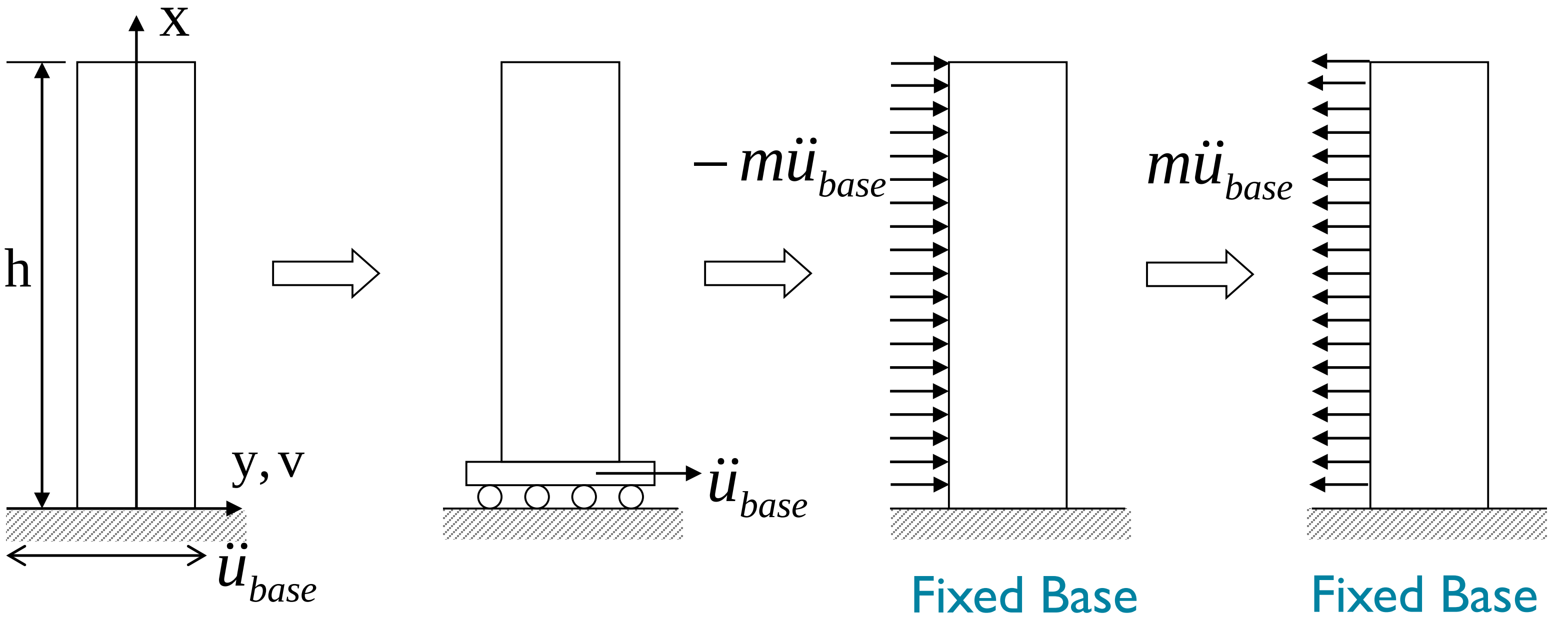
$$v(x,0)=v_0(x)=\sum_{n=1}^{\infty}B_n\sin\frac{(2n-1)\pi x}{2h}$$

$$\text{Therefore } B_n=\frac{2}{h}\int_0^h v_0(x)\sin\frac{(2n-1)\pi x}{2h}dx$$

$$\dot{v}(x,0)=\dot{v}_0(x)=\frac{(2n-1)\pi}{2h}\sqrt{\frac{GA_r}{m}}\sin\frac{(2n-1)\pi x}{2h}A_n$$

$$\text{Therefore } A_n=\frac{4}{(2n-1)\pi}\sqrt{\frac{m}{GA_r}}\int_0^h \dot{v}_0(x)\sin\frac{(2n-1)\pi x}{2h}dx$$

For an earthquake base motion at some depth



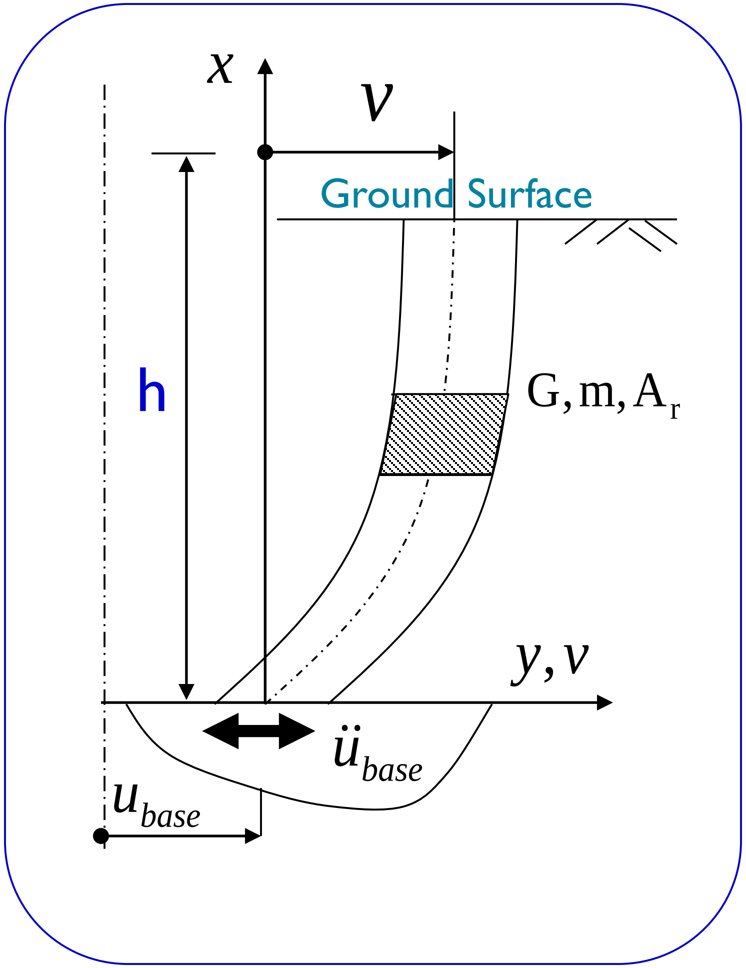
$$p(t) = -m\ddot{u}_{base}(t)$$

$$v(x,t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)x}{2h}\left[\frac{1}{\omega_n}\int_0^t \ddot{u}_{base}(\tau)\sin\omega_n(t-\tau)d\tau\right]$$

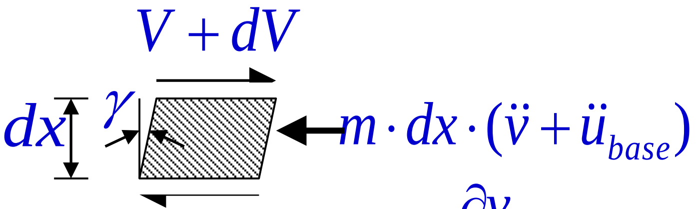
Let $\lambda_n = \dfrac{2h}{(2n-1)\pi}$, then

$$v(x,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\lambda_n\sin\frac{x}{\lambda_n}\left[\frac{1}{\omega_n}\int_0^t \ddot{u}_{base}(\tau)\sin\omega_n(t-\tau)d\tau\right]$$

Earthquake Response Analysis
Modal Solution



Dynamic Equilibrium



$$V = \tau A_r = A_r G \frac{\partial v}{\partial x}$$

$$GA_r \frac{\partial^2 v}{\partial x^2} = m\left(\frac{\partial^2 v}{\partial t^2} + \ddot{u}_{base}(t)\right)$$

$$-GA_r \frac{\partial^2 v}{\partial x^2} + m\frac{\partial^2 v}{\partial t^2} = -m\ddot{u}_{base}(t)$$

Modal Solution

$$v(x,t)=\sum_{n=1}^{\infty}X_n(x)q_n(t)$$

$q_n(t):$ *Generalized* coordinates

$$X_n(x)=\sin\frac{(2n-1)\pi x}{2h}$$

$$\sum_{n=1}^{\infty}\left[\ddot{q}_n(t)+\omega_n^2 q_n(t)\right]X_n=-\ddot{u}_{base}(t)$$

Multiplying by any mode, for instance $X_l(x)$ and integrating leads to

$$\sum_{n=1}^{\infty}\left[\ddot{q}_n(t)+\omega_n^2 q_n(t)\right]\int_0^h X_n(x)X_l(x)\,dx=-\int_0^h \ddot{u}_{base}(t)X_l(x)\,dx$$

From the condition of modal orthogonality

$$\int_0^h X_n(x) X_l(x)\, dx = 0 \qquad n \neq l$$

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = -\alpha_n \ddot{u}_{base}(\mathrm{t}) \qquad \text{SDOF - type Equation}$$

where,

$$\alpha_n = \frac{\int_0^h X_n(x)\, dx}{\int_0^h X_n^2(x)\, dx} \qquad \text{Modal Participation Factor}$$

$$= \frac{4}{(2n-1)\pi} \qquad \left[\frac{4}{\pi}, \frac{4}{3\pi}, \frac{4}{5\pi}, \frac{4}{7\pi}, \ldots\ldots\right]$$

Now, viscous modal damping can be conveniently included:

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = -\alpha_n \ddot{u}_{base}(t) \qquad \text{where } n = 1, 2, 3, \ldots.$$

For any value of n, the modal equation above can now be solved numerically as a single degree of freedom system, resulting in the modal "amplitudes" (variation of each q_n with time during the Earthquake), and the relative displacement along the beam depth is defined by:

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x) q_n(t) \quad \text{and} \quad X_n(x) = \sin\frac{(2n-1)\pi x}{2h}$$

Note: Relative velocity is obtained by replacing $q_n(t)$ by $\dot{q}_n(t)$

Note: Relative acceleration is obtained by replacing $q_n(t)$ by $\ddot{q}_n$

with the total or absolute acceleration at any depth equal to the relative $+\ddot{u}_{base}(t)$

Note: In the above, *x*= *h* corresponds to the ground surface and *x* = *0* is at the shear beam base (where the input base shaking acceleration is imparted).

Note: We typically get a very good approximation of shear beam response just by including the contribution of the first few modes (i.e., n = 1 or n = 1 and 2 or n = 1 and 2 and 3 for instance):

As such, a first mode solution would be: $v(x,t) = X_1(x) q_1(t)$

A second mode solution would be: $v(x,t) = X_2(x) q_2(t)$

and a solution based on the first 2 modes would be: $v(x,t) = X_1(x) q_1(t) + X_2(x) q_2(t)$

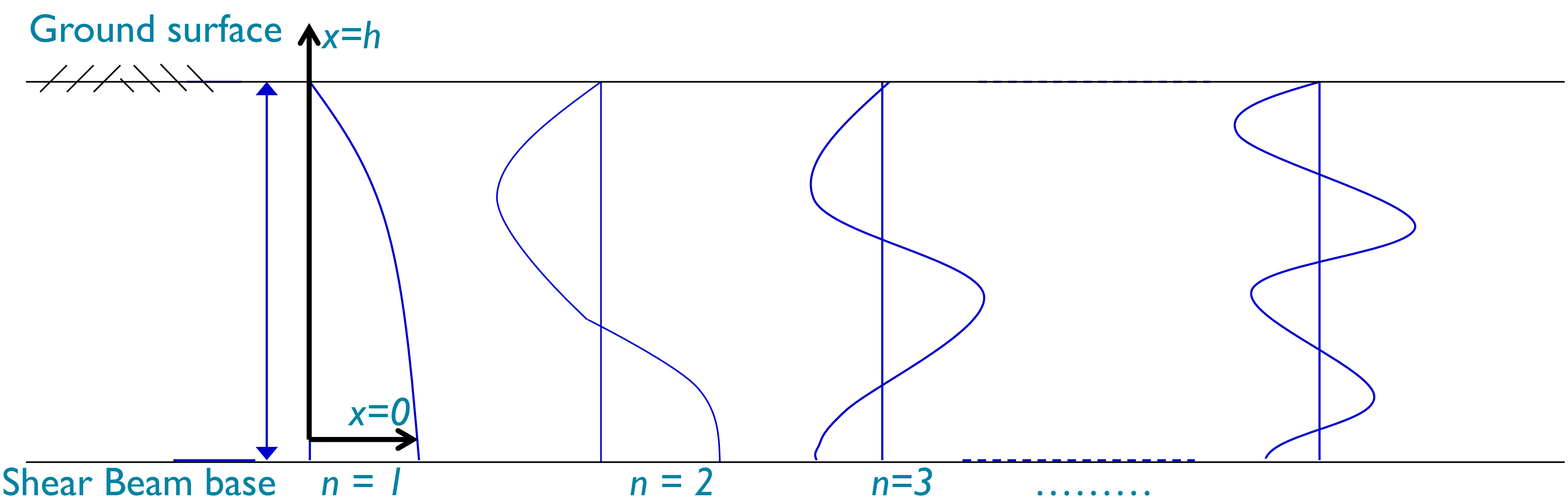
(with velocity and acceleration calculated as described above)

In the above, shear strain *g (x,t)* anywhere along the depth can be evaluated by differentiating the mode shapes to give:

$$\partial v(x,t)/\partial x = (\sum_{n=1}^{\infty} \partial X_n(x)/\partial x) q_n(t)$$

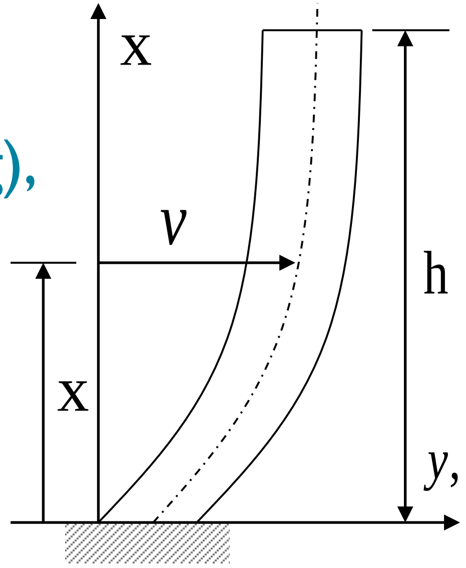
in which: $$\partial X_n(x)/\partial x = \frac{(2n-1)\pi}{2h}\cos\frac{(2n-1)\pi x}{2h}$$ (see Fig. below)

and shear stress τ *(x,t)* = *G* γ *(x,t)*



Damping in Continuous Systems

1. Fundamental Approach
   Postulate material behavior and/or mechanism and investigate the consequences



2. Ad Hoc Approach (Modal Damping), where

$$v(x,t) = \sum_{n=1}^{\infty} V_n(x)\, q_n(t)$$

and

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = p_n(t) \qquad \text{where } n = 1, 2, 3, \ldots.$$

Damping in a Shear Beam

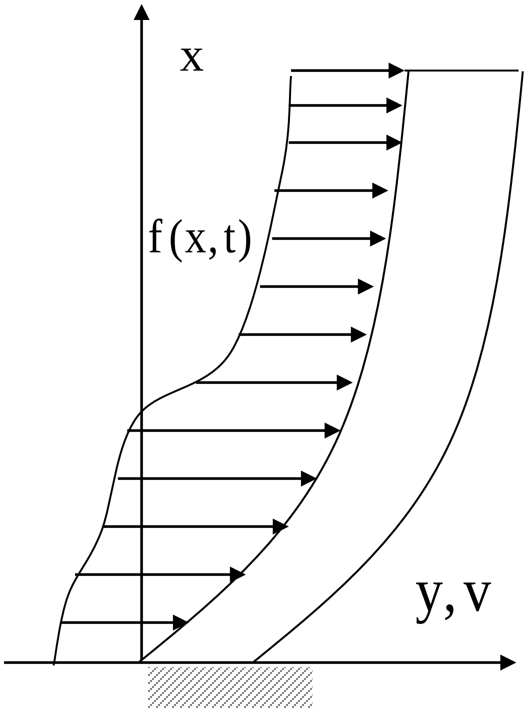
1. Equation of Motion for Forced Vibration

$$k'GA\frac{\partial^2 v}{\partial x^2} + f(x,t) = m\frac{\partial^2 v}{\partial t^2} = \rho A\frac{\partial^2 v}{\partial t^2}$$

$$let \quad v(x,t) = \sum_{n=1}^{\infty} V_n(x)\, q_n(t)$$

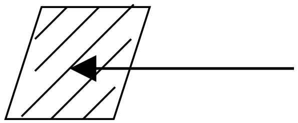
$$\text{where} \quad V_n(x) = \sin\frac{(2n-1)\pi x}{2h}, \quad \text{n} = 1, 2, 3, \ldots$$

$$and \quad \omega_n = \frac{(2n-1)\pi}{2h}\sqrt{\frac{k'G}{\rho}}, \qquad \text{n} = 1, 2, 3, \ldots$$



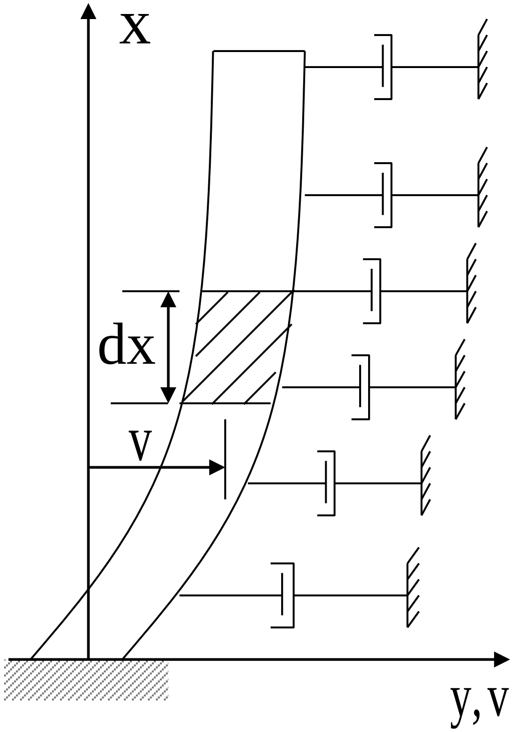
I. For External or Absolute Damping (Mass proportional)

$$f(x,t) = -F_D$$



$$F_D = -c_1 A \frac{\partial v}{\partial t} dx$$

Force per unit length



Thus, the equation of motion becomes

$$k' G A \frac{\partial^2 v}{\partial x^2} - c_1 A \frac{\partial v}{\partial t} - \rho A \frac{\partial^2 v}{\partial t^2} = 0$$

Solution can be obtained by: $v(x,t)=\sum_{n=1}^{\infty} V_n(x)\, q_n(t)$

Substituting the above results in:

$$\sum_{n=1}^{\infty} -\frac{(2n-1)^2\pi^2}{(2h)^2} q_n(t)\frac{G}{\rho}\sin\frac{(2n-1)\pi x}{2h} - \sum_{n=1}^{\infty}\frac{c_1}{\rho}\sin\frac{(2n-1)\pi x}{2h}\dot{q}_n(t) - \sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi x}{2h}\ddot{q}_n(t)=0$$

$$\sum_{n=1}^{\infty}\left(\ddot{q}_n(t)+\frac{c_1}{\rho}\dot{q}_n(t)+\omega_n^2 q_n(t)\right)\sin\frac{(2n-1)\pi x}{2h}=0$$

or $\quad \ddot{q}_n(t)+\frac{c_1}{\rho}\dot{q}_n(t)+\omega_n^2 q_n(t)=0, \quad n=1,2,3,\ldots$

Now replacing $\frac{c_1}{\rho}$ by $2\omega_n\xi_n, \quad \xi_n = \text{n}^{\text{th}}$ damping ratio

or $\xi_n = \frac{c_1}{2\rho}\frac{2h}{(2n-1)\pi}\sqrt{\frac{\rho}{G}} = \frac{c_1 h}{\sqrt{\rho G}}\frac{1}{(2n-1)}, \quad n=1,2,3,\ldots$

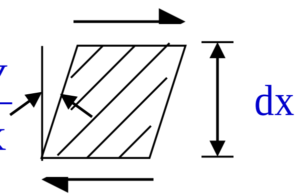
Thus, the "absolute" damping factors go to zero as n increases. If $\xi_1=\xi, \quad \xi_2=\xi/3, \quad \xi_3=\xi/5$

"This is not so good for many physical systems"

II. Strain Rate or Relative Damping (stiffness proportional)

$$F_D + dF_D = c_2 A_r \frac{\partial^2 v}{\partial x \partial t} + c_2 A_r \frac{\partial v}{\partial x}\left(\frac{\partial^2 v}{\partial x \partial t}\right) dx$$

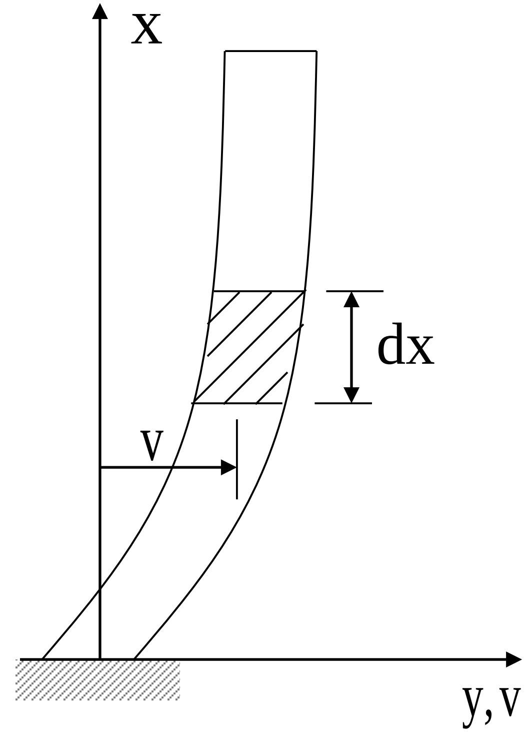
$$\gamma_{xy} = \frac{\partial v}{\partial x}$$



$$F_D = c_2 A_r \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right)$$

$c_2 =$ *damping* constant

$A_r = k' A$



Thus, $$f(x,t)dx = c_2 A_r \frac{\partial^3 v}{\partial t \partial x^3} dx$$

Equation of Motion:

$$k' G A \frac{\partial^2 v}{\partial x^2} + c_2 A_r \frac{\partial^3 v}{\partial t \partial x^2} - \rho A \frac{\partial^2 v}{\partial t^2} = 0$$

$$v(x,t) = \sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi x}{2h} q_n(t)$$

$$\sum_{n=1}^{\infty}\left[-\frac{(2n-1)^2\pi^2}{(2h)^2}\frac{G}{\rho}q_n(t) - \frac{c_2}{\rho}\frac{(2n-1)^2\pi^2}{(2h)^2}\dot{q}_n(t) - \ddot{q}_n(t)\right]\sin\frac{(2n-1)\pi x}{2h} = 0$$

$$\sum_{n=1}^{\infty}\left(\ddot{q}_n(t) + \frac{c_2}{G}\omega_n^2\dot{q}_n(t) + \omega_n^2 q_n(t)\right)\sin\frac{(2n-1)\pi x}{2h} = 0$$

$$\therefore \quad \ddot{q}_n(t) + \frac{c_2}{G}\omega_n^2\dot{q}_n(t) + \omega_n^2 q_n(t) = 0, \quad n = 1, 2, 3, \ldots$$

$$\text{Let } \frac{c_2}{G}\omega_n^2 = 2\omega_n\xi_n$$

$$\Rightarrow \xi_n = \frac{c_2}{2G}\frac{(2n-1)\pi}{2h}\sqrt{\frac{G}{\rho}} = \frac{c_2\pi}{4h\sqrt{G\rho}}(2n-1), \quad n = 1, 2, 3, \ldots \quad \text{"Relative Damping"}$$

$$\text{if } \xi_1 = \xi, \ \xi_2 = 3\xi, \ \xi_3 = 5\xi,$$

- i.e., “Damping increases rapidly”
- Rayleigh Damping → a combination of both.

III. Arbitrary Damping

Take the damping ratio ξ_n= a set of arbitrary constants. Then substitute in the equation of motion:

$$\ddot{q}_n(t) + 2\xi_n \omega_n^2 \dot{q}_n(t) + \omega_n^2 q_n(t) = -\alpha_n \ddot{u}_g(t)$$

<u>Summary</u>	"Damping"
1. Absolute Mass Proportional Damping	$\xi_n = \frac{c_1}{2\omega_n \rho}, \quad n = 1, 2, 3, \ldots$
2. Relative Stiffness Proportional Damping	$\xi_n = \frac{c_2 \omega_n}{2G}, \quad n = 1, 2, 3, \ldots$
3. Rayleigh Damping	$\xi_n = \frac{c_1}{2\omega_n \rho} + \frac{c_2 \omega_n}{2G}, \quad n = 1, 2, 3, \ldots$
4. Constant Damping in all Modes	$\xi_n = \text{const.}, \quad n = 1, 2, 3, \ldots$
5. Damping as a set of Arbitrary Constants	$\xi_n = \text{Arbitrary const.}, \quad n = 1, 2, 3, \ldots$

Earthquake "Response Spectrum Technique"

$$v(x,t)=\sum_{n=1}^{\infty}V_n(x)\,q_n(t), \qquad \ddot{q}_n(t)+2\xi_n\omega_n^2\dot{q}_n(t)+\omega_n^2 q_n(t)=-\alpha_n\ddot{u}_g(t)$$

<u>Solution</u> "Duhamel Integral"

$$q_n(t)=-\frac{\alpha_n}{\omega_n}\left[\int_0^t \ddot{u}_g(\tau)e^{\xi_n\omega_n(t-\tau)}\sin\omega_n\sqrt{1-\xi_n^2}(t-\tau)d\tau\right]$$

Denoting the integral by the symbol $V_n(t)$, i.e.,

$$V_n(t)=\int_0^t \ddot{u}_g(\tau)e^{\xi_n\omega_n(t-\tau)}\sin\omega_n\sqrt{1-\xi_n^2}(t-\tau)d\tau \qquad \text{Thus} \qquad q_n(t)=-\frac{\alpha_n}{\omega_n}V_n(t)$$

$$q_n(t)\Big|_{max}=-\frac{\alpha_n}{\omega_n}V_n(t)\Big|_{max}=-\frac{\alpha_n}{\omega_n}S_v, \qquad S_v \text{ is the spectral velociy}$$

Note that the spectral displacement $S_d=\frac{S_v}{\omega_n}$ and the spectral acceleration $S_a=S_v\omega_n$

$$q_n(t)\Big|_{max}=-\alpha_n S_d, \qquad \text{and}$$

$$v_i(x,t)\Big|_{max}=-X_i(x)q_i(t)\Big|_{max}, \qquad i=1,2,3,.....$$

Displacement at the Top

At the top of the beam, i.e., at x=h, $v_i(h,t) = \left|X_i(h)\right| \frac{\alpha_i}{\omega_i} S_v, \quad i = 1, 2, 3, ...$

Thus for a single mode, $v_i(h,t) = \frac{\alpha_i}{\omega_i} S_v = \frac{4S_v}{\pi(2i-1)\omega_1(2i-1)} = \frac{4S_v}{\pi(2i-1)^2 \omega_1}$

Combine according to sum of absolute values

$$v(h,t)_{max} \le v_1(h,t)_{max} + v_2(h,t)_{max} + v_3(h,t)_{max} + v_4(h,t)_{max} + ...$$

Combine according to the root mean square (RMS)

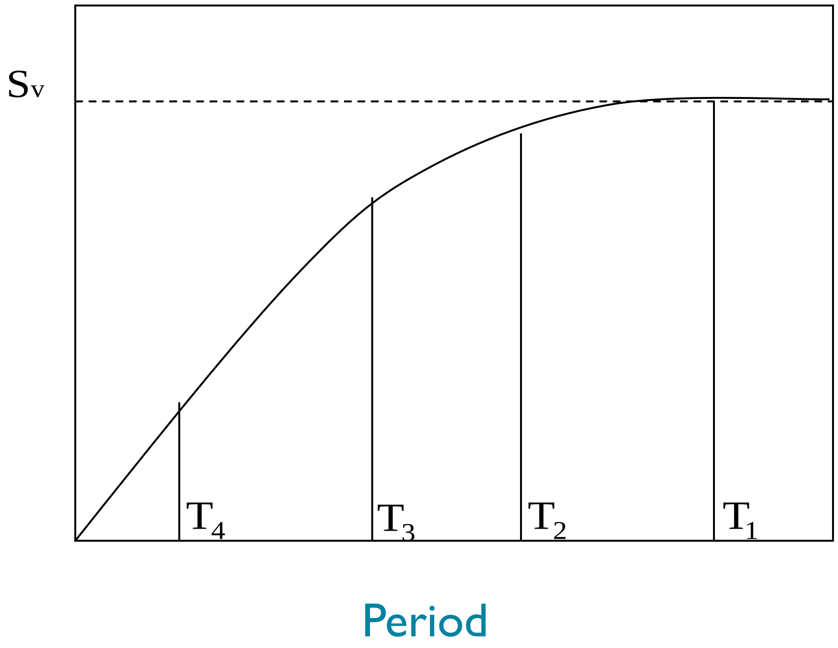
$$v(h,t)_{max} \cong \sqrt{v_1^2(h,t)_{max} + v_2^2(h,t)_{max} + v_3^2(h,t)_{max} + v_4^2(h,t)_{max} + ...}$$

Substituting the above values (and simply just assuming the same S_v for all frequencies)

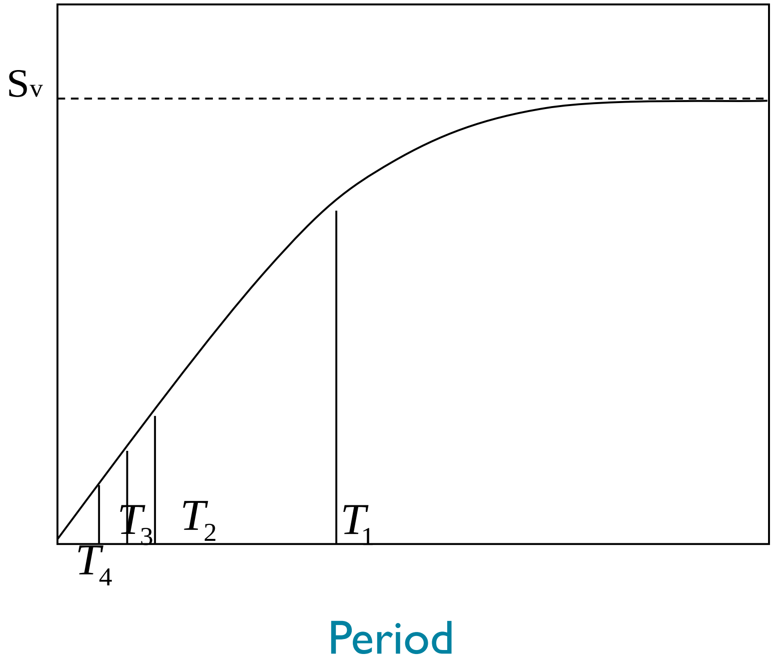
$$v(h,t)_{max} \le \frac{4S_v}{\pi\omega_1}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + ...\right) = \frac{1.575}{\omega_1} S_v$$

$$v(h,t)_{max} \cong \frac{4S_v}{\pi\omega_1}\left(1 + \frac{1}{81} + \frac{1}{625} + ...\right)^{1/2} = \frac{1.280}{\omega_1} S_v$$

The first mode contributes significantly to the displacement response (on the average, with S_v assumed to be constant for all modes)



Tall System



Short System

Acceleration at the Top

$$\ddot{v}(x,t) = \sum_{i=1}^{\infty} X_i(x)\,\ddot{q}_i(t)$$

Total acceleration $$a(x,t) = \ddot{v}(x,t) + \ddot{u}_{base}(t)$$

but (see next slide) $$\ddot{u}_{base}(t) = \sum_{i=1}^{\infty} \alpha_i \ddot{u}_{base}(t) X_i(x)$$

So $$a(x,t) = \sum_{i=1}^{\infty} \left(\ddot{q}_i(t) + \alpha_i \ddot{u}_{base}(t)\right) X_i(x)$$

For the i[th] mode $$a_i(x,t) = \left(\ddot{q}_i(t) + \alpha_i \ddot{u}_{base}(t)\right) X_i(x) = \left(-\omega_i^2 q_i(t)\right) X_i(x)$$

$$a_i(h,t)\big|_{\max} = \alpha_i \omega_i S_v = \frac{4}{(2i-1)\pi}\omega_1 (2i-1) S_v = \frac{4\omega_1 S_v}{\pi}$$

Combining as before $$a(h,t)\big|_{\max} \le \frac{4\omega_1 S_v}{\pi}\left[1+1+1+1+\ldots\right]$$

$$a(h,t)\big|_{\max} \cong \frac{4\omega_1 S_v}{\pi}\left[1+1+1+1+\ldots\right]^{1/2}$$

Thus, with the assumed constant S_v , the modes on the average, tend to contribute equally to the acceleration at the top.

Section to show that $\sum_{i=1}^{\infty} \alpha_i X_i(x) = 1$

Let $\sum_{n=1}^{\infty} b_n X_n(x) = 1$ where b_n are constants to be identified

multiply by X_l and integrate over the height

$$\sum_{n=1}^{\infty} b_n \int_0^h X_n(x) X_l(x)\, dx = \int_0^h X_l(x)\, dx$$

Therefore, on account of modal orthogonality

$$b_n = \frac{\int_0^h X_n(x)\, dx}{\int_0^h X_n^2(x)\, dx} = \alpha_n$$

(the Modal Participation Factor)

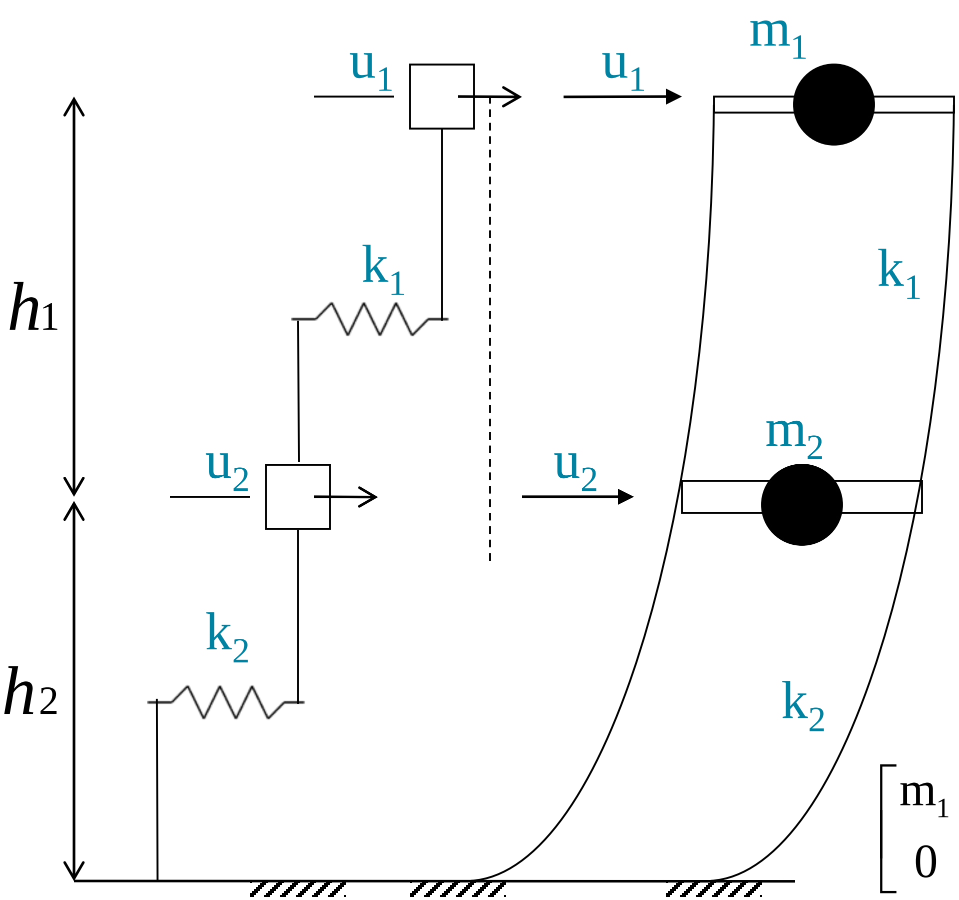
where $\alpha_n = \dfrac{4}{(2n-1)\pi}$ $\left[\dfrac{4}{\pi}, \dfrac{4}{3\pi}, \dfrac{4}{5\pi}, \dfrac{4}{7\pi}, \ldots\ldots\right]$

Numerical Implementation of Shear Beam Model

Allows for handling of Stratified soil profiles (i.e., layers of different stiffness/damping properties)

Allows for implementation of nonlinear hysteretic soil shear stress-strain properties

Example: 2-DOF Soil shear beam , with h_i = distance between masses



$$m_1 = \rho_1 h_1/2$$

$$m_2 = (\rho_1 h_1 + \rho_2 h_2)/2$$

$$k_1 = G_1 / h_1$$

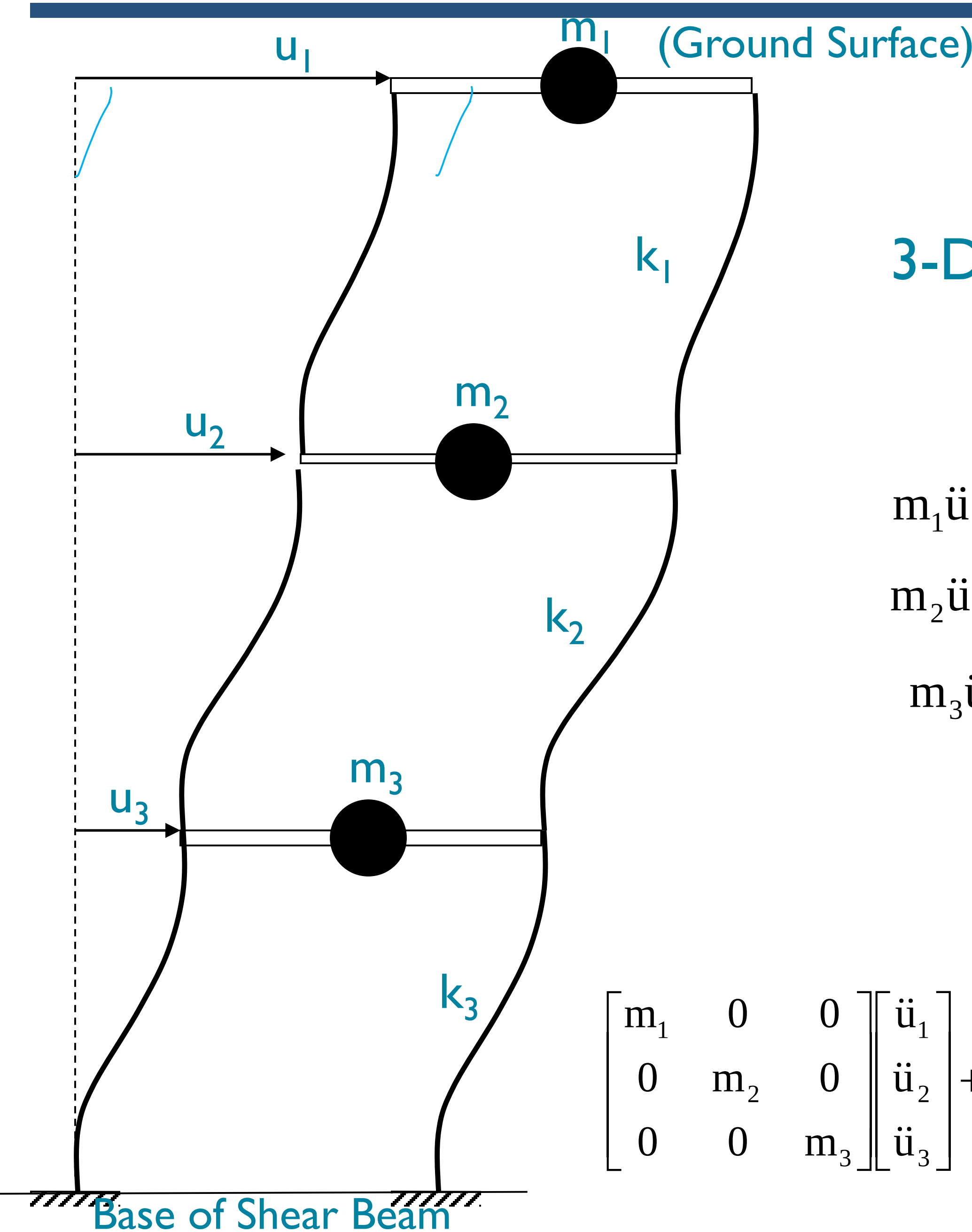
$$k_2 = G_2 / h_2$$

$$m_1\ddot{u}_1 + k_1(u_1 - u_2) = -m_1\ddot{u}_{base}$$

$$m_2\ddot{u}_2 + k_1(u_2 - u_1) + k_2 u_2 = -m_2\ddot{u}_{base}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\ddot{u}_{base}$$

or $\underline{M}\underline{\ddot{u}} + \underline{K}\underline{u} = -\underline{M}\underline{1}\ddot{u}_{base}$ where $\underline{1}$ is the Identity Matrix



3-DOF Shear Beam

$$m_1\ddot{u}_1 + k_1(u_1 - u_2) = -m_1\ddot{u}_{base}$$

$$m_2\ddot{u}_2 + k_1(u_2 - u_1) + k_2(u_2 - u_3) = -m_2\ddot{u}_{base}$$

$$m_3\ddot{u}_3 + k_2(u_3 - u_2) + k_3 u_3 = -m_3\ddot{u}_{base}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & (k_2 + k_3) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}\ddot{u}_{base}$$

For n DOF system,

$$
\underline{M} = \begin{bmatrix}
m_1 & & & & & & & & & \\
& m_2 & & & & & & & & \\
& & - & & & & & & & \\
& & & - & & & & & & \\
& & & & - & & & & & \\
& & & & & - & & & & \\
& & & & & & m_m & & & \\
& & & & & & & - & & \\
& & & & & & & & - & \\
& & & & & & & & & m_n
\end{bmatrix}
$$

$$
\underline{K} = \begin{bmatrix}
(k_1) & -k_1 & 0 & - & - & - & - & - & - & 0 \\
-k_1 & (k_1 + k_2) & -k_2 & 0 & - & - & - & - & - & 0 \\
0 & & - & & & & & & & \\
0 & & & - & & & & & & \\
| & & & & - & & & & & \\
| & & & & & - & & & & \\
0 & & & & 0 & -k_m & (k_m + k_{m+1}) & -k_{m+1} & 0 & \\
| & & & & & & & - & & 0 \\
| & & & & & & & & - & -k_{n-1} \\
0 & 0 & - & - & - & - & - & 0 & -k_{n-1} & (k_{n-1} + k_n)
\end{bmatrix}
$$

In Matrix Equation form,

$$
\begin{bmatrix}
m_1 &&&&&&&&& \\
& m_2 &&&&&&&& \\
&& - &&&&&&& \\
&&& - &&&&&& \\
&&&& - &&&&& \\
&&&&& - &&&& \\
&&&&&& m_k &&& \\
&&&&&&& - && \\
&&&&&&&& - & \\
&&&&&&&&& m_n
\end{bmatrix}
\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ | \\ | \\ | \\ | \\ \ddot{u}_m \\ | \\ | \\ \ddot{u}_n \end{bmatrix}
+
\begin{bmatrix}
(k_1) & -k_1 & 0 & - & - & - & - & - & - & 0 \\
-k_1 & (k_1+k_2) & -k_2 & 0 & - & - & - & - & - & 0 \\
0 & & - & & & & & & & \\
0 & & & - & & & & & & \\
| & & & & - & & & & & \\
| & & & & & - & & & & \\
0 & & & & 0 & -k_m & (k_m+k_{m+1}) & -k_{m+1} & 0 & \\
| & & & & & & & - & & 0 \\
| & & & & & & & & - & -k_{n-1} \\
0 & 0 & - & - & - & - & - & 0 & -k_{n-1} & (k_{n-1}+k_n)
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ | \\ | \\ | \\ | \\ u_m \\ | \\ | \\ u_n \end{bmatrix}
= -
\begin{bmatrix} m_1 \\ m_2 \\ | \\ | \\ | \\ | \\ m_m \\ | \\ | \\ m_n \end{bmatrix}
\ddot{u}_{base}
$$

Note:

1) A viscous damping Matrix can be added in the form of:

$$\mathbf{c} = a_0\,\mathbf{m} + a_1\,\mathbf{k}$$

2) Viscous damping can alternatively be specified at the element by element level

For a Nonlinear system, the matrix equation may be written in the form:

$$
\begin{bmatrix}
m_1 & & & & & & & & \\
& m_2 & & & & & & & \\
& & - & & & & & & \\
& & & - & & & & & \\
& & & & - & & & & \\
& & & & & - & & & \\
& & & & & & m_k & & \\
& & & & & & & - & \\
& & & & & & & & - \\
& & & & & & & & & m_n
\end{bmatrix}
\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ | \\ | \\ | \\ | \\ \ddot{u}_m \\ | \\ | \\ \ddot{u}_n \end{bmatrix}
+
\begin{bmatrix} F_1 \\ F_2 \\ | \\ | \\ | \\ | \\ F_m \\ | \\ | \\ F_n \end{bmatrix}
= -
\begin{bmatrix} m_1 \\ m_2 \\ | \\ | \\ | \\ | \\ m_m \\ | \\ | \\ m_n \end{bmatrix}
\ddot{u}_{base}
$$

Numerical Solution of Equation of Motion

Average Acceleration Method (Trapezoidal method)

$m\,a + c\,v + k\,d = f\,(t)$

In the above mass-spring-dashpot (damper) equation of motion:

f (t) is the forcing function defined as given f_i values at (t_i) in which i = 0, 1, 2, … NTS (number of time steps), with

the time step between any t_i and t_{i+1} equal to Δt, and

m, c, k are the mass, damping and stiffness matrices.

Initial Conditions: $d\,(t = 0) = d_0$, and $v\,(t = 0) = v_0$

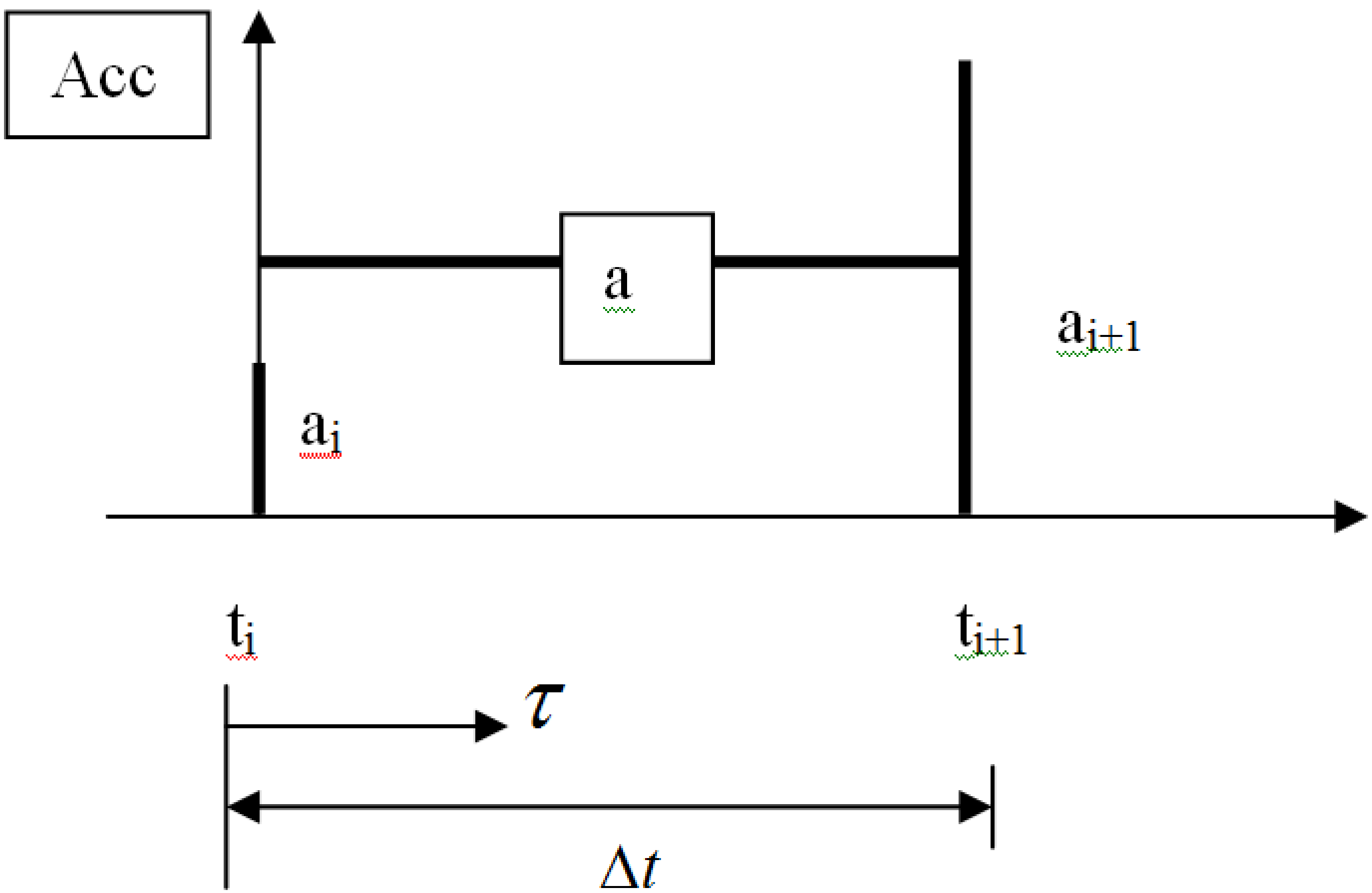
From these conditions and the known f_0, one can find $a\,(t = 0) = a_0$ from the Equation above

At any time step $t = t_{i+1}$: $m\,a_{i+1} + c\,v_{i+1} + k\,d_{i+1} = f_{i+1}$ (Eq. 1)

Now, we need to find a_{i+1}, v_{i+1}, and d_{i+1}, using f_{i+1}, and information from the previous time step (i.e., a_i, v_i, and d_i)

Average acceleration dictates that (see figure):

$$a = (a_{i+1} + a_i) / 2 \qquad \text{(Eq. 2)}$$



Integrate to get velocity

$v = v_i + t (a_{i+1} + a_i) / 2$ (Eq. 3)

Integrate above to get displacement:

$d = d_i + v_i\ t + (t^2 /4) (a_{i+1} + a_i)$
(Eq. 4)

At the end of the Interval, $t = \Delta t$ and therefore (using Eqs. 3 and 4)

$v_{i+1} = v_i + (\Delta t/ 2) (a_{i+1} + a_i)$ (Eq. 5)

$d_{i+1} = d_i + v_i\ \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)$ (Eq. 6)

Now, substitute Equations 5 and 6 into Equation 1 to get

$m\ a_{i+1} + c (v_i + (\Delta t/ 2) (a_{i+1} + a_i)) + k (d_i + v_i\ \Delta t + (\Delta t^2 / 4) (a_{i+1} + a_i)) = f_{i+1}$
or,
$(m + c (\Delta t / 2) + k (\Delta t^2 / 4))\ a_{i+1} = f_{i+1} – c (v_i + (\Delta t/ 2) a_i) - k (d_i + v_i\ \Delta t + (\Delta t^2 / 4)a_i)$ (Eq. 7)

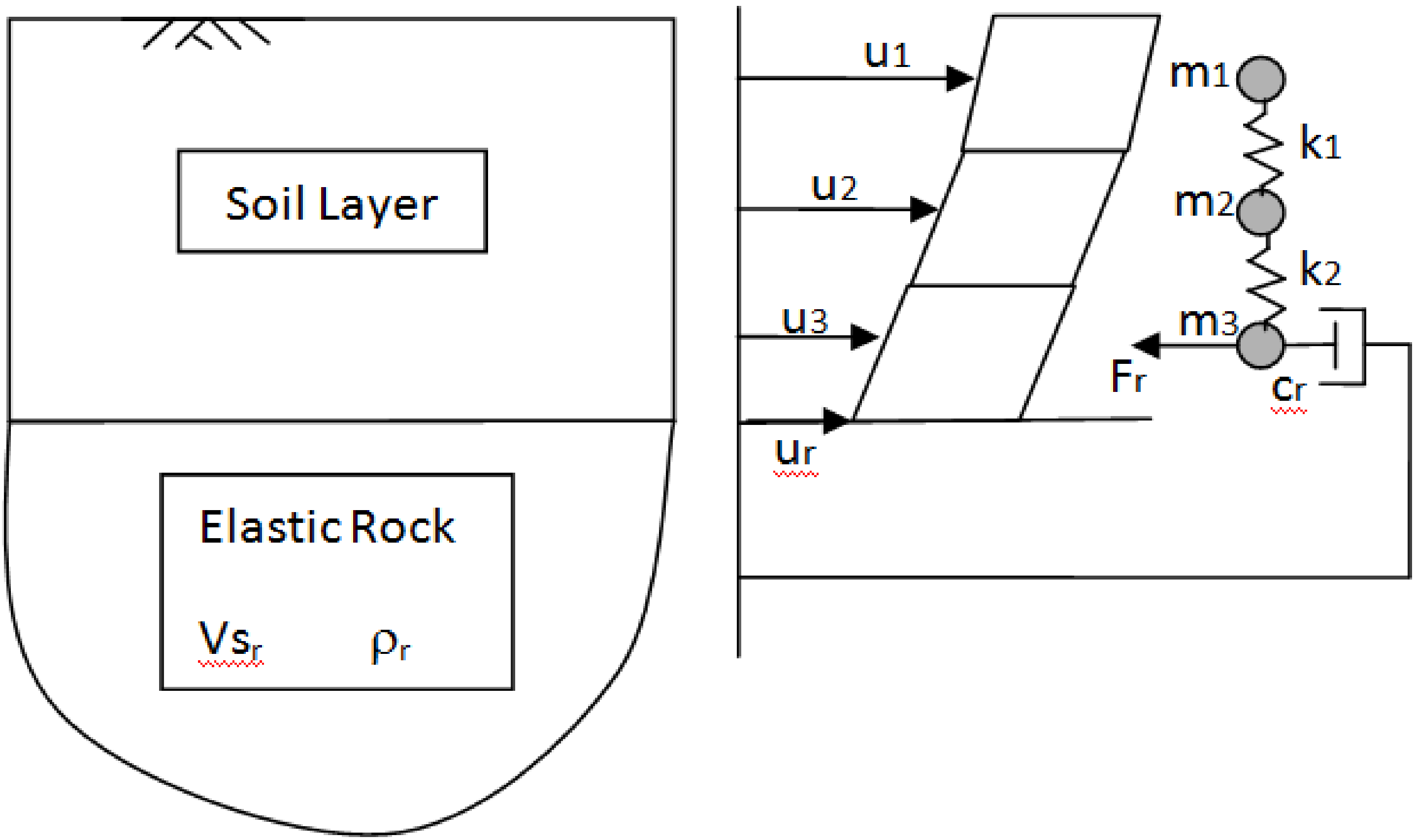
where $[m + c (\Delta t / 2) + k (\Delta t^2 / 4)]$ is known as the effective mass = m*) and a matrix inversion process is involved for multi-degree-of-freedom cases.

Solve Eq. 7 for a_{i+1}, and (using Eqs. 4 and 5) solve for v_{i+1} , d_{i+1}

Now all quantities are known at i +1 and we are ready to go to the next time step (repeat the above procedure).

Site Amplification: Shear Beam on Elastic Rock (The "Lysmer" Dashpot approach)

Illustration using 3-Degree of freedom model



Lysmer, J. (1978). "Analytical procedures in soil dynamics," Report No. UCB/EERC-78/29, University of California at Berkeley, Earthquake Engineering Research Center, Richmond, CA.

Lysmer, J. and Kuhlemeyer, A.M. (1969). "Finite dynamic model for infinite media," J.of the En Mechanics Division, ASCE, 95, 859-877.

Important note: In this formulation, u^t denotes Total Displacement (i.e. displacement of the *r*ock base u_r + displacement of the soil stratum relative to the rock). As such, the matrix equation of motion can be defined by *(for any time step t)*:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{u}_1^{\,t} \\ \ddot{u}_2^{\,t} \\ \ddot{u}_3^{\,t} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{32} & c_{33}+c_r \end{bmatrix} \begin{bmatrix} \dot{u}_1^{\,t} \\ \dot{u}_2^{\,t} \\ \dot{u}_3^{\,t} \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1^{\,t} \\ u_2^{\,t} \\ u_3^{\,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_r \end{bmatrix}$$

where the **c** matrix (except for the term c_r) is formed according to the usual Rayleigh damping approach in which $\mathbf{c} = a_o\, \mathbf{m} + a_1\, \mathbf{k}$

The underlying elastic rock stratum introduces the terms c_r and F_r defined by:

$c_r = \rho_r V s_r$ (rock mass density multiplied by rock shear wave velocity)

$F_r = 2\, \rho_r V s_r\, \dot{u}_{ir}$

In the above, $\dot{u}_{ir}$ is *velocity* of the incident rock motion (often taken for simplicity as ½ that of an available (or assumed) nearby **recorded** rock-outcrop motion).

Implementation notes:

1) If needed, filter out the superfluous low frequency components of the input $\ddot{u}_g$ rock-outcrop acceleration time history (say anything below 0.3 Hz). This will help in avoiding superfluous drift upon integrating this motion to get rock velocity $\dot{u}_r$ and double integrating to get rock displacement u_r. After integration, you can plot the rock velocity and displacement to check for drift.

2) Divide the rock-outcrop velocity values by ½ throughout to obtain the motion that will be used in the matrix equation above (i.e., $\dot{u}_{ir} = ½\ \dot{u}_r$). Using $\dot{u}_{ir}$, go ahead and define the input time history F_r according to the above equation.

3) Solve the matrix equation in time using the usual implicit time integration scheme (average acceleration method, or linear acceleration method for instance), and store the output displacement, velocity, and acceleration vectors.

4) The soil absolute (total) acceleration $\ddot{\boldsymbol{u}}^t$ is now already available for plotting. To obtain the soil relative displacement and the soil relative velocity vectors (relative to the rock base motion), you must subtract the corresponding rock motion time histories first (as mentioned in the note above, the matrix equation solves for total motion).